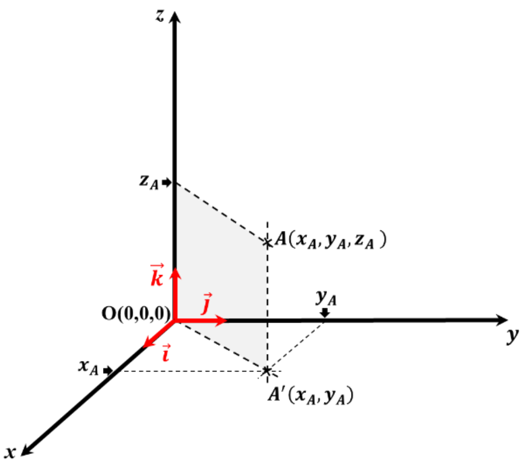
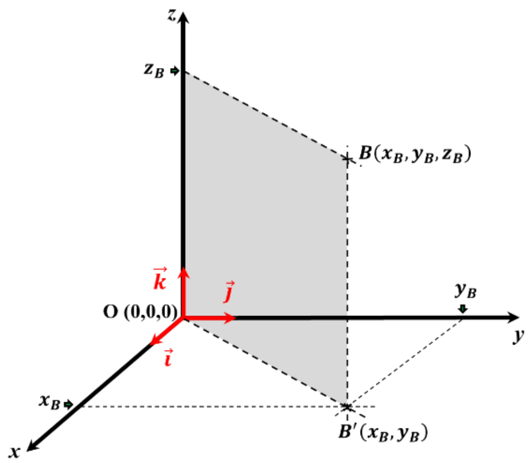
Examples are force ($\vec{F}$), displacement ($\overrightarrow{AB}$) and velocity ($\vec{V}$). A vector is represented by an arrow at the given angle. The head of the arrow indicates the sense, and the length usually represents the magnitude of the vector.

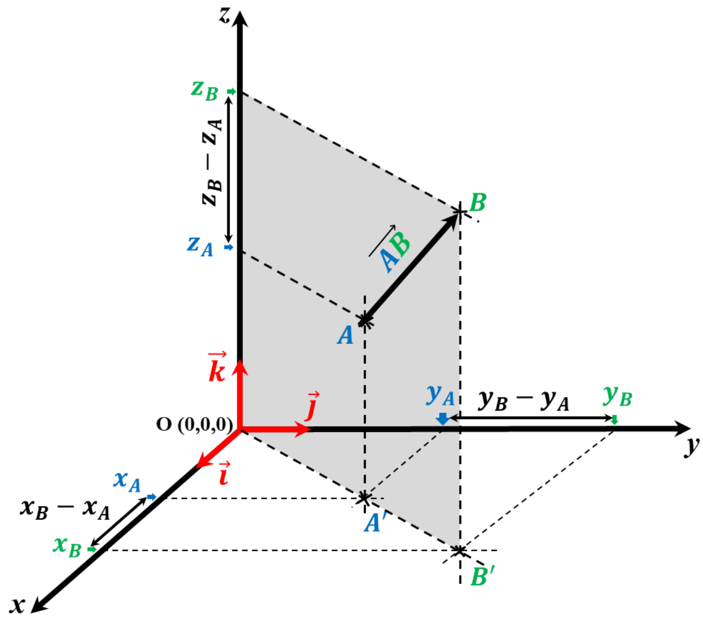
Vectors are usually described in terms of their components in a coordinate system. It means that when working with vectors in mathematics or physics, they are often represented or characterized by specifying their individual parts (components) within a particular coordinate system. In a three-dimensional space, this coordinate system might be represented as $(\vec{\imath},\vec{\jmath},\vec{k})$ or (x, y, z). Each component indicates the vector's magnitude along a specific direction within that coordinate system, allowing for a comprehensive description of the vector's properties.

Example: It is required to represent $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, that are two particles' positions, in an orthogonal basis $(Oxyz)$ and then draw the vector $\overrightarrow{AB}$. It is assumed that their components are positive and that $(x_A < x_B, y_A < y_B, z_A < z_B)$. We start by representing the positions A and B :





Both points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ must be represented in the same orthogonal basis (Oxyz) to draw the vector $(\overrightarrow{AB})$, and it appropriately introduces the following figure for further illustration.



The vector $(\overrightarrow{AB})$ can be written as follows:

$$\overrightarrow{AB} = (x_B - x_A)\,\vec{i} + (y_B - y_A)\,\vec{j} + (z_B - z_A)\,\vec{k}$$

If we consider $\vec{V} = \overrightarrow{AB}$, we obtain :

$$\vec{V} = \overrightarrow{AB} = (x_B - x_A)\,\vec{i} + (y_B - y_A)\,\vec{j} + (z_B - z_A)\,\vec{k}$$

Knowing that :

$$\vec{V} = x_V\,\vec{i} + y_V\,\vec{j} + z_V\,\vec{k}$$

Hence, the components of $\vec{V}$ are :

$$x_V = x_B - x_A$$
$$y_V = y_B - y_A$$
$$z_V = z_B - z_A$$

Its magnitude is :

$$\|\vec{V}\| = \sqrt{x_V^2 + y_V^2 + z_V^2} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

a. The unit vectors

A unit vector is typically denoted as follows :

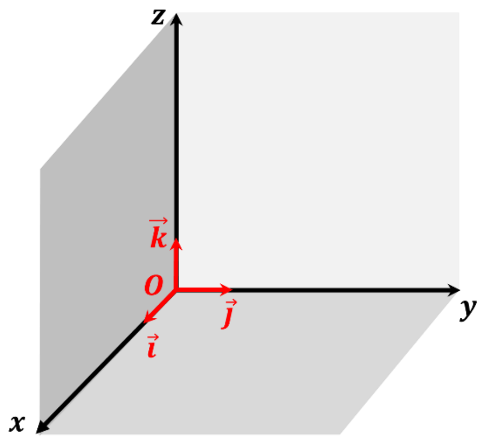
- ✓ A unit vector has a magnitude of 1 (eg. $\|\vec{U}\| = 1$)
- ✓ Unit vectors are often used to describe the direction of a given vector
- ✓ A unit vector is a vector of unit magnitude used to specify a particular spatial direction

In three-dimensional space, the unit vectors along the *x*, *y*, and *z* axes are correspondingly designated as :

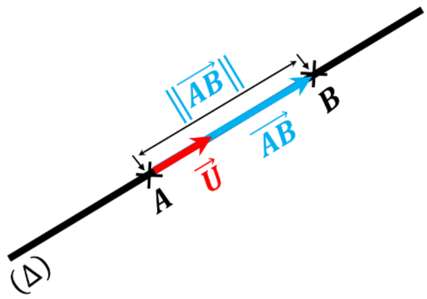
i. $\vec{\imath}$: along the *x*-axis and it has components (1, 0, 0)

ii. $\vec{\jmath}$: along the *y*-axis and it has components (0, 1, 0)

iii. $\vec{k}$: along the *y*-axis and it has components (0, 0, 1)



These unit vectors are essential in describing the direction of other vectors in three-dimensional space.



The formula of the vector $\overrightarrow{AB}$ is :

$$\overrightarrow{AB} = \|\overrightarrow{AB}\|.\vec{U}$$

To create a unit vector in a specific direction, you divide the vector in that direction by its magnitude. Note that the magnitude of this unit vector effectively normalizes its length to 1 $\left(\|\vec{U}\| = 1\right)$.

$$\vec{U} = \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}$$

Where $\left(\|\overrightarrow{AB}\|\right)$ is the magnitude of the vector $\left(\overrightarrow{AB}\right)$ and $\left(\vec{U}\right)$ is the unit vector associated with $\left(\overrightarrow{AB}\right)$.

b. Operations on vectors

Several arithmetic operations can be performed on vectors, including addition, subtraction and multiplication. Consider the two following vectors $\vec{V}$ and $\vec{G}$ where:

$$\vec{V} = x_V \vec{\imath} + y_V \vec{\jmath} + z_V \vec{k}$$

$$\vec{G} = x_G \vec{\imath} + y_G \vec{\jmath} + z_G \vec{k}$$

- **Addition of two vector**

$$\vec{V} + \vec{G} = (x_V + x_G)\,\vec{\imath} + (y_V + y_G)\,\vec{\jmath} + (z_V + z_G)\,\vec{k}$$

- **Subtraction of two vectors**

$$\vec{V} - \vec{G} = (x_V - x_G)\,\vec{\imath} + (y_V - y_G)\,\vec{\jmath} + (z_V - z_G)\,\vec{k}$$

- **Multiplication**
    - **Scalar multiplication** $(\lambda : constant)$

$$\lambda.\overrightarrow{V} = \lambda.\left(x_V \vec{\imath} + y_V \vec{\jmath} + z_V \vec{k}\right)$$

$$\lambda.\overrightarrow{V} = \lambda.x_V \vec{\imath} + \lambda.y_V \vec{\jmath} + \lambda.z_V \vec{k}$$

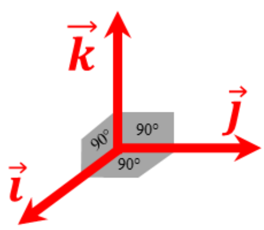
    - **Dot Product (Scalar Product)**

$$\vec{V} \cdot \vec{G} = \|\vec{V}\| \cdot \|\vec{G}\| \cdot \cos\theta$$

Remark:

$\vec{\imath} \cdot \vec{\imath} = \vec{\jmath} \cdot \vec{\jmath} = \vec{k} \cdot \vec{k} = 1 \ (\theta = 0^{\circ} \Rightarrow \cos 0 = 1)$

$\vec{\imath} \cdot \vec{\jmath} = \vec{\imath} \cdot \vec{k} = \vec{\jmath} \cdot \vec{k} = 0 \ (\theta = 90^{\circ} \Rightarrow cos\, 90 = 0)$



The dot product (scalar product) between $\vec{V}$ and $\vec{K}$ can be developed as shown bellow:

$$\vec{V}\cdot\vec{G} = \left(x_V\vec{\imath} + y_V\vec{\jmath} + z_V\,\vec{k}\right).\left(x_G\vec{\imath} + y_G\,\vec{\jmath} + z_G\,\vec{k}\right)$$

$$\vec{V}\cdot\vec{G} = (x_V\cdot x_G)\overbrace{\vec{\imath}\cdot\vec{\imath}}^{1} + (x_V\cdot y_G)\overbrace{\vec{\imath}\cdot\vec{\jmath}}^{0} + (x_V\cdot z_G)\overbrace{\vec{\imath}\cdot\vec{k}}^{0} + (y_V\cdot x_G)\overbrace{\vec{\jmath}\cdot\vec{\imath}}^{0} + (y_V\cdot y_G)\overbrace{\vec{\jmath}\cdot\vec{\jmath}}^{1}$$
$$+ (y_V\cdot z_G)\underbrace{\vec{\jmath}\cdot\vec{k}}_{0} + (z_V\cdot x_G)\underbrace{\vec{k}\cdot\vec{\imath}}_{0} + (z_V\cdot y_G)\underbrace{\vec{k}\cdot\vec{\jmath}}_{0} + (z_V\cdot z_G)\underbrace{\vec{k}\cdot\vec{k}}_{1}$$

The final result is :

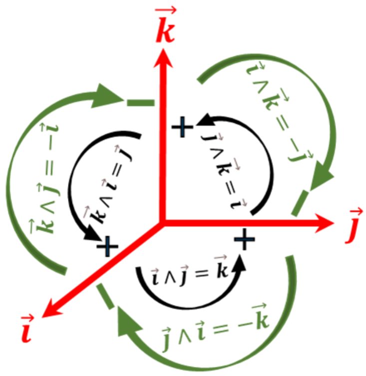
$$\vec{V}\cdot\vec{G} = (x_V\cdot x_G) + (y_V\cdot y_G) + (z_V\cdot z_G)$$

- **Cross Product (Vector Product)**

$$\left|\vec{V}\wedge\vec{G}\right| = \|\vec{V}\|\cdot\|\vec{G}\|\cdot\sin\theta$$

Remark:

$$\vec{\imath}\wedge\vec{\imath} = \vec{\jmath}\wedge\vec{\jmath} = \vec{k}\wedge\vec{k} = 0\ (\theta = 0^\circ \Rightarrow\ sin\,0 = 0)$$
$$\vec{\imath}\wedge\vec{\jmath} = \vec{k}\,,\ \vec{\jmath}\wedge\vec{k} = \vec{\imath}\,,\ \vec{k}\wedge\vec{\imath} = \vec{\jmath}$$
$$\vec{\imath}\wedge\vec{k} = -\vec{\jmath}\,,\ \vec{\jmath}\wedge\vec{\imath} = -\vec{k}\,,\ \vec{k}\wedge\vec{\jmath} = -\vec{\imath}$$



The cross product (vector product) between $\vec{V}$ and $\vec{K}$ can be developed as shown bellow:

$$\vec{V}\wedge\vec{G} = \left(x_V\vec{\imath} + y_V\vec{\jmath} + z_V\,\vec{k}\right)\wedge\left(x_G\vec{\imath} + y_G\,\vec{\jmath} + z_G\,\vec{k}\right)$$

$$\vec{V}\wedge\vec{G} = (x_V\cdot x_G)\underbrace{\vec{\imath}\wedge\vec{\imath}}_{=\mathbf{0}} + (x_V\cdot y_G)\underbrace{\vec{\imath}\wedge\vec{\jmath}}_{=\vec{k}} + (x_V\cdot z_G)\underbrace{\vec{\imath}\wedge\vec{k}}_{=-\vec{\jmath}} + (y_V\cdot x_G)\underbrace{\vec{\jmath}\wedge\vec{\imath}}_{=-\vec{k}}$$
$$+ (y_V\cdot y_G)\underbrace{\vec{\jmath}\wedge\vec{\jmath}}_{=\mathbf{0}} + (y_V\cdot z_G)\underbrace{\vec{\jmath}\wedge\vec{k}}_{=\vec{\imath}} + (z_V\cdot x_G)\underbrace{\vec{k}\wedge\vec{\imath}}_{=\vec{\jmath}} + (z_V\cdot y_G)\underbrace{\vec{k}\wedge\vec{\jmath}}_{=-\vec{\imath}}$$
$$+ (z_V\cdot z_G)\underbrace{\vec{k}\wedge\vec{k}}_{=\mathbf{0}}$$

$$\vec{V}\wedge\vec{G} = (x_V\cdot y_G)\,\vec{k} + (x_V\cdot z_G)\,(-\vec{\jmath}) + (y_V\cdot x_G)\,(-\vec{k}) + (y_V\cdot z_G)\,\vec{\imath} + (z_V\cdot x_G)\,\vec{\jmath}$$
$$+ (z_V\cdot y_G)\,(-\vec{\imath})$$

$$\vec{V} \wedge \vec{G} = (y_V \cdot z_G)\,\vec{\imath} + (z_V \cdot y_G)\,(-\vec{\imath}) + (z_V \cdot x_G)\,\vec{\jmath} + (x_V \cdot z_G)\,(-\vec{\jmath}) + (x_V \cdot y_G)\,\vec{k} + (y_V \cdot x_G)\,(-\vec{k})$$

$$\vec{V} \wedge \vec{G} = (y_V \cdot z_G - z_V \cdot y_G)\,\vec{\imath} + (x_V \cdot z_G - z_V \cdot x_G)\vec{\jmath} + (x_V \cdot y_G - y_V \cdot x_G)\,\vec{k}$$

This result can be found using the determinant method as follows:

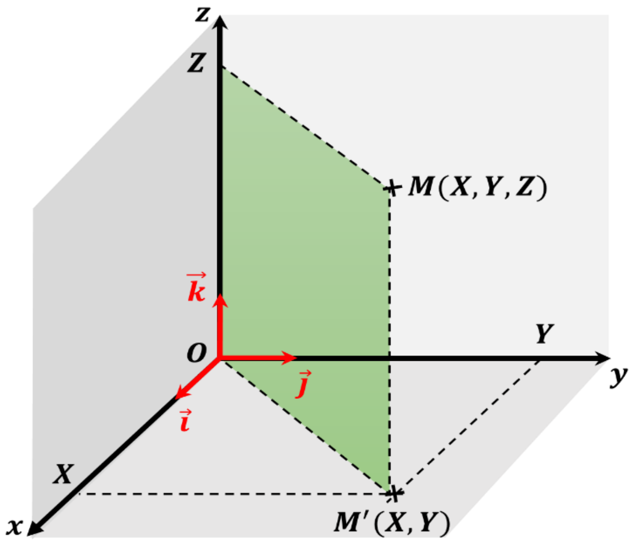
$$\vec{V} \wedge \vec{G} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_V & y_V & z_V \\ x_G & y_G & z_G \end{vmatrix}$$

$$\vec{V} \wedge \vec{G} = (y_V \cdot z_G - z_V \cdot y_G)\,\vec{\imath} - (x_V \cdot z_G - z_V \cdot x_G)\,\vec{\jmath} + (x_V \cdot y_G - y_V \cdot x_G)\,\vec{k}$$

I.1.2.4. Coordinate systems

A coordinate system is a mathematical framework used to represent the positions of points or objects in space, whether in two-dimensional or three-dimensional settings. It provides a way to describe the location of points (particle/moving-object) using numerical values called coordinates.

Example: The illustration in the ensuing figure depicts the representation of $M(X, Y, Z)$ in the orthogonal basis $(Oxyz)$. Here, (X, Y, Z) represent the components of *M*, while $(\vec{\imath}, \vec{\jmath}, \vec{k})$ denote the unit vectors associated with the axes (*Ox, Oy, Oz*), respectively.

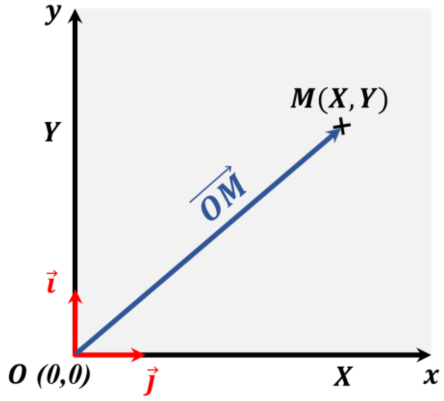


In the ensuing sections, the physical quantities such as position vector $(\overrightarrow{OM})$, velocity $(\vec{V})$ and acceleration $(\vec{a})$ will be expressed in the various types of coordinate systems:

A. Cartesian Coordinate System:

i. 2D Cartesian Coordinate System:

In the 2D Cartesian coordinate system, points are located using two numerical values: x and y. The point (0,0) is called the origin, and it is the intersection of the x-axis (horizontal) and the y-axis (vertical).



- ✓ Position vector ($\overrightarrow{OM}$) :

$$\overrightarrow{OM} = X\,\vec{\imath} + Y\,\vec{\jmath}$$

$$\|\overrightarrow{OM}\| = \sqrt{X^2 + Y^2}$$

- ✓ Velocity vector ($\vec{V}$) :

$$\vec{V} = V_x\,\vec{\imath} + V_y\,\vec{\jmath}$$

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(X\,\vec{\imath} + Y\,\vec{\jmath})$$

$$\vec{V} = \frac{dX}{dt}\,\vec{\imath} + \frac{dY}{dt}\,\vec{\jmath} = X'\,\vec{\imath} + Y'\,\vec{\jmath}$$

$$V_x = \frac{dX}{dt} = X' \qquad \text{and} \qquad V_y = \frac{dY}{dt} = Y'$$

$$\|\vec{V}\| = \sqrt{V_x^2 + V_y^2}$$

- Acceleration vector $(\vec{a})$:

$$\vec{a} = a_x\,\vec{\imath} + a_y\,\vec{\jmath}$$

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d\vec{V}}{dt} = \frac{d^2}{dt^2}(X\,\vec{\imath} + Y\,\vec{\jmath}) = \frac{d}{dt}\left(V_x\,\vec{\imath} + V_y\,\vec{\jmath}\right)$$

$$\vec{a} = \frac{d^2X}{dt^2}\,\vec{\imath} + \frac{d^2Y}{dt^2}\,\vec{\jmath} = \frac{dV_x}{dt}\,\vec{\imath} + \frac{dV_y}{dt_{\ y}}\,\vec{\jmath}$$

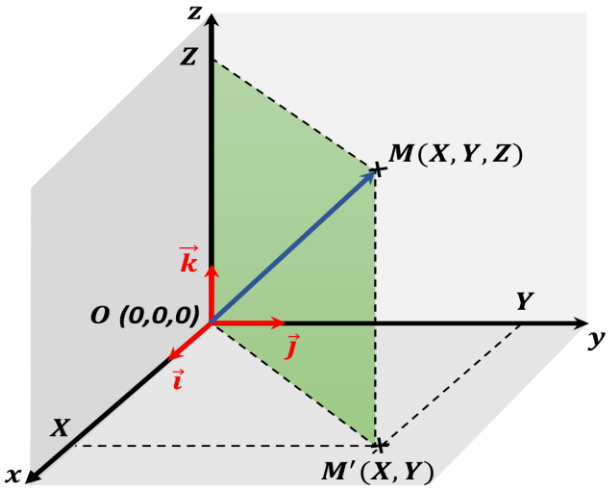
$$\vec{a} = X''\,\vec{\imath} + Y''\,\vec{\jmath} = V_x'\,\vec{\imath} + V_y'\,\vec{\jmath}$$

$$a_x = X'' = V_x' \qquad \text{and} \qquad a_y = Y'' = V_y'$$

$$\|\vec{a}\| = \sqrt{a_x^2 + a_y^2}$$

ii. 3D Cartesian Coordinate System:

The 3D Cartesian coordinate system extends the 2D system by adding a third dimension, the z-axis (vertical, extending in and out of the plane defined by the x and y axes).



- Position vector $(\overrightarrow{OM})$:

$$\overrightarrow{OM} = X\,\vec{\imath} + Y\,\vec{\jmath} + Z\,\vec{k}$$

$$\|\overrightarrow{OM}\| = \sqrt{X^2 + Y^2 + Z^2}$$

✓ Velocity vector ($\vec{V}$) :

$$\vec{V} = V_x\,\vec{\imath} + V_y\,\vec{\jmath} + V_z\,\vec{k}$$

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}\left(X\,\vec{\imath} + Y\,\vec{\jmath}\ + Z\,\vec{k}\right)$$

$$\vec{V} = \frac{dX}{dt}\,\vec{\imath} + \frac{dY}{dt}\,\vec{\jmath}\ + \frac{dZ}{dt}\vec{k} = X'\,\vec{\imath} + Y'\,\vec{\jmath} + Z'\,\vec{k}$$

$$V_x = \frac{dX}{dt} = X' \qquad V_y = \frac{dY}{dt} = Y' \qquad V_z = \frac{dZ}{dt} = Z'$$

$$\|\vec{V}\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

✓ Acceleration vector ($\vec{a}$) :

$$\vec{a} = a_x\,\vec{\imath} + a_y\,\vec{\jmath} + a_z\vec{k}$$

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d\vec{V}}{dt} = \frac{d^2}{dt^2}\left(X\,\vec{\imath} + Y\,\vec{\jmath} + Z\,\vec{k}\right) = \frac{d}{dt}\left(V_x\,\vec{\imath} + V_y\,\vec{\jmath} + V_z\vec{k}\right)$$

$$\vec{a} = \frac{d^2X}{dt^2}\,\vec{\imath} + \frac{d^2Y}{dt^2}\,\vec{\jmath} + \frac{d^2Z}{dt^2}\,\vec{k} = \frac{dV_x}{dt}\,\vec{\imath} + \frac{dV_y}{dt_y}\,\vec{\jmath} + \frac{dV_z}{dt}\,\vec{k}$$

$$\vec{a} = X''\,\vec{\imath} + Y''\,\vec{\jmath}\ + Z''\,\vec{k} = V_x'\,\vec{\imath} + V_y'\,\vec{\jmath}\ +\ V_z'\,\vec{k}$$

$$a_x = X'' = V_x' \qquad a_y = Y'' = V_y' \qquad a_z = Z'' = V_z'$$

$$\|\vec{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Example:

The position vector ($\overrightarrow{OM}$) of a moving particle is written as follows:

$$\overrightarrow{OM} = (t^3 + 2t)\,\vec{\imath} + 3t^2\,\vec{\jmath} + 3t\,\vec{k}$$

a. Represent ($\overrightarrow{OM}$) in the orthogonal basis (*Oxyz*) and find its magnitude at $t = 1\ sec$

b. Find the velocity vector ($\vec{V}$) and its magnitudes $\|\vec{V}\|$ at $t = 1\ sec$

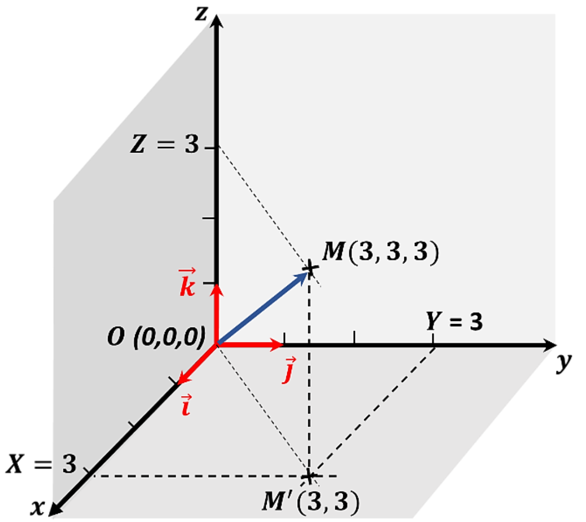
c. Find the acceleration vector ($\vec{a}$) and its magnitudes $\|\vec{a}\|$ at $t = 1\ sec$

Solution:

a. At $t = 1\ sec$ the formula of the position vector ($\overrightarrow{OM}$) is :

$$\overrightarrow{OM}(t = 1\ sec) = (1^2 + 2 \times 1)\,\vec{\imath} + (3 \times 1^2)\,\vec{\jmath} + (3 \times 1)\,\vec{k}$$

$$\overrightarrow{OM}(t = 1\ sec) = 3\,\vec{\imath} + 3\,\vec{\jmath} + 3\,\vec{k}$$



The position vector magnitude is: $\|\overrightarrow{OM}\| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{27} = 5.2\ m$

b. Velocity:

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}\left((t^3 + 2t)\,\vec{\imath} + 3t^2\,\vec{\jmath} + 3t\,\vec{k}\right)$$

$$\vec{V} = \frac{d}{dt}\,(t^3 + 2t)\,\vec{\imath} + \frac{d}{dt}\,(3t^2)\,\vec{\jmath} + \frac{d}{dt}(3t)\,\vec{k} = (3t^2 + 2)\,\vec{\imath} + 6t\,\vec{\jmath} + 3\,\vec{k}$$

Then : $\vec{V} = (3t^2 + 2)\,\vec{\imath} + 6t\,\vec{\jmath} + 3\,\vec{k}$, at $t = 1\ sec$: $\vec{V} = 5\,\vec{\imath} + 6\,\vec{\jmath} + 3\,\vec{k}$

The velocity magnitude is: $\|\vec{V}\| = \sqrt{5^2 + 6^2 + 3^2} = \sqrt{70} = 8.37\ m.sec^{-1}$

a. Acceleration:

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left((3t^2 + 2)\,\vec{\imath} + 6t\,\vec{\jmath} + 3\,\vec{k}\right)$$

$$\vec{V} = \frac{d}{dt}\,(3t^2 + 2)\,\vec{\imath} + \frac{d}{dt}\,(6t)\,\vec{\jmath} + \frac{d}{dt}(3)\,\vec{k} = (6t)\,\vec{\imath} + 6\,\vec{\jmath}$$

Then : $\vec{a} = (6t)\,\vec{\imath} + 6\,\vec{\jmath}$, at $t = 1\ sec$: $\vec{a} = 6\,\vec{\imath} + 6\,\vec{\jmath}$

The acceleration magnitude is: $\|\vec{a}\| = \sqrt{6^2 + 6^2} = \sqrt{36} = 6\ m.sec^{-2}$