Example

A particle moves in cylindrical coordinates according to the following functions of time:

$$\rho(t) = 2t$$
$$\theta(t) = \omega t \qquad (\omega = cte)$$
$$z(t) = 3t^2$$

1. Find the velocity vector $\vec{V}$ in cylindrical coordinates.
2. Determine the speed $(\|\vec{V}\|)$ of the particle as a function of time.
3. Find the acceleration vector $\vec{a}$ in cylindrical coordinates.
4. Determine the magnitude of the acceleration $\|\vec{a}\|$ as a function of time.

Solution

1. Find the velocity $\vec{V}$: $\quad \vec{V} = \dfrac{d\overrightarrow{OM}}{dt}$

Given : $\quad \overrightarrow{OM} = \rho \cdot \overrightarrow{U_\rho} + z\vec{k} \Rightarrow \vec{V} = \dfrac{d}{dt}(\rho \cdot \overrightarrow{U_\rho} + z\vec{k})$

$$\vec{V} = \left(\frac{d\rho}{dt}\right) \cdot \overrightarrow{U_\rho} + \rho \cdot \left(\frac{d\overrightarrow{U_\rho}}{dt}\right) + \left(\frac{dz}{dt}\right)\vec{k}$$

Knowing that:

$$\dot{\rho} = \left(\frac{d\rho}{dt}\right) = \frac{d}{dt}(2t) = 2 \; and \; \dot{z} = \left(\frac{dz}{dt}\right) = \frac{d}{dt}(3t^2) = 6t$$

And :
$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d}{dt}(\omega t) = \omega$$
$$\left(\frac{d\overrightarrow{U_\rho}}{dt}\right) = \dot{\overrightarrow{U_\rho}} = \dot{\theta}\overrightarrow{U_\theta} = \omega.\overrightarrow{U_\theta}$$

Hence : $\quad \vec{V} = 2\,\overrightarrow{U_\rho} + 2\omega t\,\overrightarrow{U_\theta} + 6t\,\vec{k}$

2. Determining the speed (V) (the magnitude of the velocity vector):

$$\|\vec{V}\| = \sqrt{2^2 + (2\omega t)^2 + (6t)^2} = \sqrt{4 + 4\omega^2 t^2 + 36t^2} = \sqrt{4(\omega^2 + 9)t^2 + 4}$$

Thus : $\quad \vec{V} = 2\sqrt{(\omega^2 + 9)t^2 + 1}$

3. Find the acceleration vector $\vec{a}$: $\quad \vec{a} = \dfrac{d\vec{V}}{dt} = \dfrac{d^2\overrightarrow{OM}}{dt^2}$

As expressed previously, the acceleration vector $(\vec{a})$, in cylindrical coordinates, can be articulated as:

$$\vec{a} = \left(\ddot{\rho} - \rho \cdot \dot{\theta}^2\right) \overrightarrow{U_\rho} + \left(\rho \cdot \ddot{\theta} + 2\,\dot{\rho} \cdot \dot{\theta}\right) \overrightarrow{U_\theta} + \ddot{z}\vec{k}$$

Determining the values of $\dot{\rho}, \ddot{\rho}, \dot{\theta}, \dot{\theta}^2, \ddot{\theta}$ and $\ddot{z}$ permits the obtention of the final expression of the acceleration vector $(\vec{a})$:

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{d}{dt}(2t) = 2 \Rightarrow \ddot{\rho} = \frac{d^2\rho}{dt^2} = \frac{d\dot{\rho}}{dt} = \frac{d}{dt}(2) = 0$$

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d}{dt}(\omega t) = \omega \Rightarrow \ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{d\dot{\theta}}{dt} = \frac{d}{dt}(\omega) = 0$$

$$\dot{\theta}^2 = (\omega)^2 = \omega^2$$

$$\dot{z} = \frac{dz}{dt} = \frac{d}{dt}(3t^2) = 6t \Rightarrow \ddot{z} = \frac{d^2z}{dt^2} = \frac{d\dot{z}}{dt} = \frac{d}{dt}(6t) = 6$$

Accordingly:

$$\vec{a} = (0 - (2t) \cdot (\omega)^2) \overrightarrow{U_\rho} + \left((2t) \cdot 0 + 2 \cdot 2 \cdot \omega\right) \overrightarrow{U_\theta} + 6\,\vec{k}$$

Then :

$$\vec{a} = -2t\omega^2 \cdot \overrightarrow{U_\rho} + 4\omega \cdot \overrightarrow{U_\theta} + 6\,\vec{k}$$

4. Determining the magnitude of the acceleration $\|\vec{a}\|$:

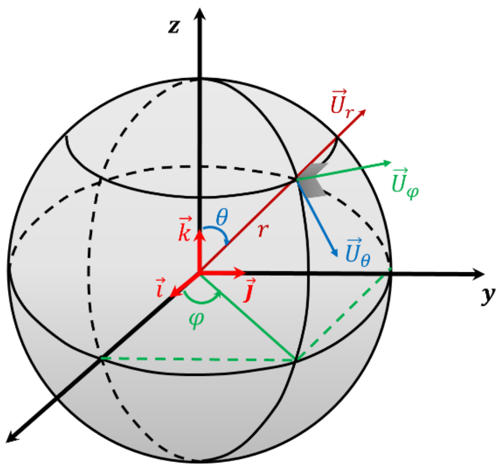
$$a = \|\vec{a}\| = \sqrt{(2\omega^2 t)^2 + (4\omega)^2 + (6)^2} = \sqrt{4\omega^4 t^2 + 16\omega^2 + 36}$$

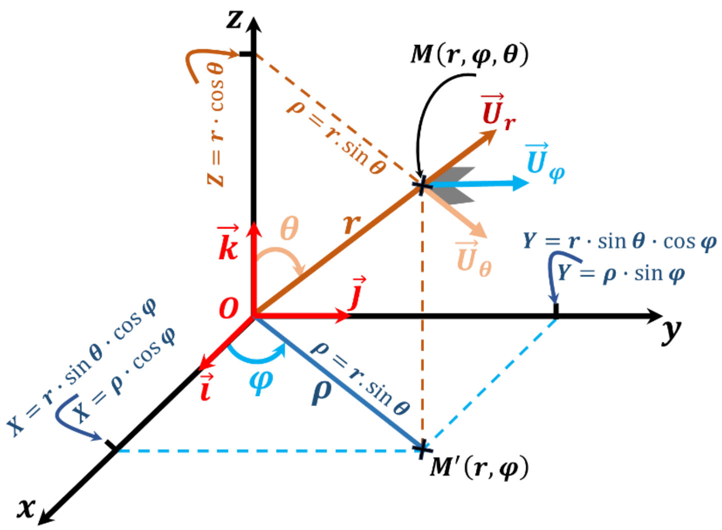
D. Spherical Coordinate System: Spherical coordinates describe a point in $3D$ space using three values: radial distance (r), polar angle (θ) and azimuthal angle (φ). It is particularly useful for describing points on the surface of a sphere. Here is a brief explanation of each coordinate:

- ✓ **Radial Distance $(\boldsymbol{r})$:** This is the straight-line distance from the origin (the point (0,0,0)) to the point (M) in space. It is sometimes denoted as (r) and is always a non-negative value $(r > 0)$.
- ✓ **Polar Angle $(\boldsymbol{\theta})$:** Also known as the zenith angle, it represents the angle measured from the positive z-axis to the line segment connecting the origin to the point. The

polar angle is usually measured in radians and ranges from 0 to π radians (180 degrees).

Azimuthal Angle (φ): Also known as the azimuth angle or the azimuth, this angle is measured in the xy-plane from the positive x-axis to the projection of the line segment onto the xy-plane. The azimuthal angle is usually measured in radians and can range from 0 to 2π radians (360 degrees).





The conversion between Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, φ) is given by the following equations:

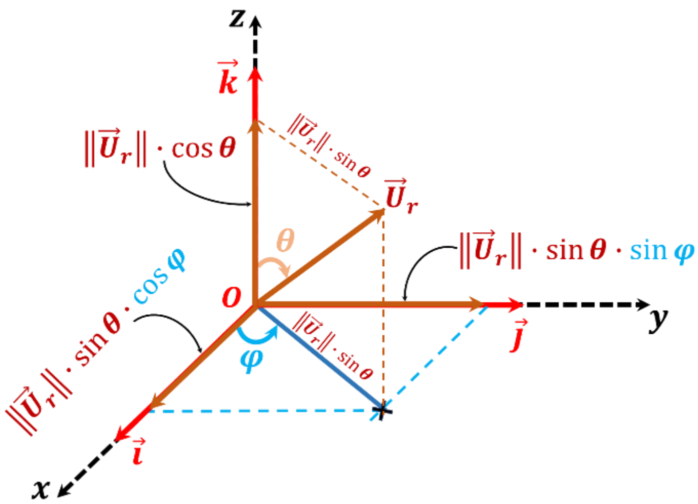
Cartesian to Spherical	Spherical to Cartesian	Interval of variation
$X = r \cdot \sin\theta \cdot \cos\varphi$	$r = \sqrt{X^2 + Y^2 + Z^2}$	$r \in [0, +\infty[$
$Y = r \cdot \sin\theta \cdot \sin\varphi$	$\tan\varphi = \frac{Y}{X}$	$0 \leq \varphi \leq 2\pi$
$Z = r \cdot \cos\theta$	$\cos\theta = \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}$	$0 \leq \theta \leq \pi$

Unit vectors $(\overrightarrow{U_r}, \overrightarrow{U_\varphi}, \overrightarrow{U_\theta})$ in spherical coordinates can be defined as follows:

- **Radial Unit Vector ($\overrightarrow{U_r}$):**

  The radial unit vector points in the direction of increasing radial distance (r) and is represented and given as follows:

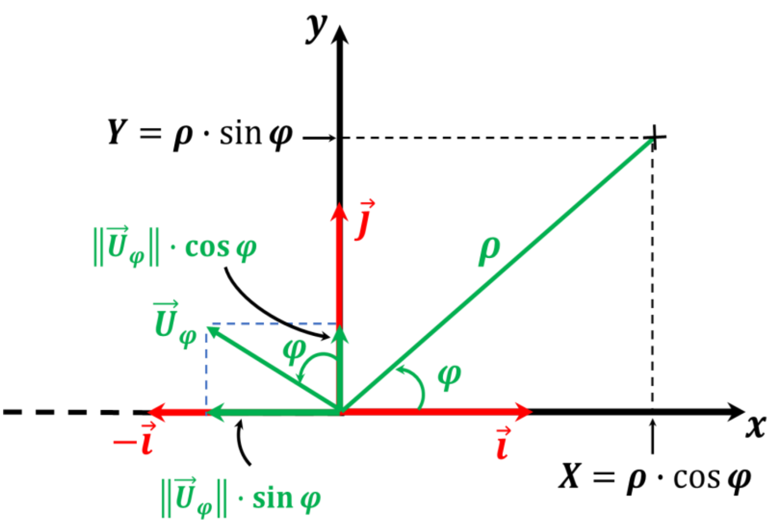
$$\overrightarrow{U_r} = \sin\theta \cdot \cos\varphi \ \vec{\imath} + \sin\theta \cdot \sin\varphi \ \vec{\jmath} + \cos\theta \ \vec{k}$$



- **Azimuthal Unit Vector ($\overrightarrow{U_\varphi}$):**

  The azimuthal unit vector points in the direction of increasing azimuthal angle (φ) and is represented and given as follows:

$$\overrightarrow{U_\varphi} = -\sin\varphi \ \vec{\imath} + \cos\varphi \ \vec{\jmath}$$



- **Polar Unit Vector ($\overrightarrow{U_\theta}$):**

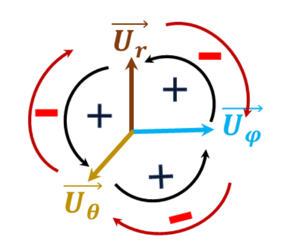
The polar unit vector points in the direction of increasing polar angle (θ). To find its expression, we apply the cross product (vector product) between the radial unit vector $\left(\overrightarrow{U_r}\right)$ and the azimuthal unit vector $\left(\overrightarrow{U_\varphi}\right)$ as follows:

$$\overrightarrow{U_\theta} = \overrightarrow{U_\varphi} \wedge \overrightarrow{U_r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cdot\cos\varphi & \sin\theta\cdot\sin\varphi & \cos\theta \end{vmatrix}$$

$$\begin{aligned}\overrightarrow{U_\theta} = &(\cos\varphi\cdot\cos\theta - (\sin\theta\cdot\sin\varphi)\times 0)\,\vec{\imath} \\ &-\big((-\sin\varphi)\cdot\cos\theta - (\sin\theta\cdot\cos\varphi)\times 0\big)\,\vec{\jmath} \\ &+\big((-\sin\varphi)\times\sin\theta\cdot\sin\varphi - \sin\theta\cdot\cos\varphi\times\cos\varphi\big)\,\vec{k}\end{aligned}$$

$$\overrightarrow{U_\theta} = \cos\varphi\cdot\cos\theta\;\vec{\imath} + \sin\varphi\cdot\cos\theta\;\vec{\jmath} - \sin\theta\,\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=\,1}\,\vec{k}$$

Hence: $$\overrightarrow{U_\theta} = \cos\varphi\cdot\cos\theta\;\vec{\imath} + \sin\varphi\cdot\cos\theta\;\vec{\jmath} - \sin\theta\;\vec{k}$$



Positive Cross product	Negative Cross product
$\overrightarrow{U_\varphi} \wedge \overrightarrow{U_r} = \overrightarrow{U_\theta}$	$\overrightarrow{U_r} \wedge \overrightarrow{U_\varphi} = -\overrightarrow{U_\theta}$
$\overrightarrow{U_r} \wedge \overrightarrow{U_\theta} = \overrightarrow{U_\varphi}$	$\overrightarrow{U_\theta} \wedge \overrightarrow{U_r} = -\overrightarrow{U_\varphi}$
$\overrightarrow{U_\theta} \wedge \overrightarrow{U_\varphi} = \overrightarrow{U_r}$	$\overrightarrow{U_\varphi} \wedge \overrightarrow{U_\theta} = -\overrightarrow{U_r}$

✓ **Position vector $\left(\overrightarrow{OM}\right)$**

The position vector $\left(\overrightarrow{OM}\right)$ in Cartesian coordinates is given as:

$$\overrightarrow{OM} = X\,\vec{\imath} + Y\;\vec{\jmath} + Z\,\vec{k}$$

Substituting the expressions of *X*, *Y* and *Z* in the last formula then the spherical coordinates of the position vector $\left(\overrightarrow{OM}\right)$ will be given by:

$$\overrightarrow{OM} = r\cdot\sin\theta\cdot\cos\varphi\,\vec{\imath} + r\cdot\sin\theta\cdot\sin\varphi\;\vec{\jmath} + r\cdot\cos\theta\;\vec{k}$$

Here, (r) is the radial distance, (θ) is the polar angle (angle from the positive *z*-axis), and (φ) is the azimuthal angle (angle in the *xy*-plane from the positive *x*-axis).

As we can see, the radial distance (r) is the common factor in the expression of the position vector. In other word, the position vector in spherical coordinates can be written as follows:

$$\overrightarrow{OM} = r \cdot (\sin\theta \cdot \cos\varphi\, \vec{\imath} + \sin\theta \cdot \sin\varphi\; \vec{\jmath} + \cos\theta\; \vec{k})$$

Given that :

$$\overrightarrow{U_r} = \sin\theta \cdot \cos\varphi\; \vec{\imath} + \sin\theta \cdot \sin\varphi\; \vec{\jmath} + \cos\theta\; \vec{k}$$

Hence,

$$\overrightarrow{OM} = r\,\overrightarrow{U_r}$$

✓ **Velocity vector $(\vec{V})$**

In spherical coordinates, the velocity vector is given as follows:

$$\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(r\overrightarrow{U_r}) = \left(\frac{dr}{dt}\right)\vec{U}_r + r \cdot \frac{d\overrightarrow{U_r}}{dt}$$

Developing this expression leads to:

$$\vec{V} = \dot{r}\,\vec{U}_r + r \cdot \frac{d}{dt}(sin\,\theta \cdot cos\,\varphi\;\vec{\imath} + sin\,\theta \cdot sin\,\varphi\;\vec{\jmath} + cos\,\theta\;\vec{k})$$

And:

$$\vec{V} = \dot{r}\,\vec{U}_r + r \cdot \left[\left(\frac{d}{dt}(\sin\theta) \cdot cos\,\varphi + sin\,\theta \cdot \frac{d}{dt}(\cos\varphi)\right)\vec{\imath} + \left(\frac{d}{dt}(sin\,\theta) \cdot sin\,\varphi + sin\,\theta \cdot \frac{d}{dt}(sin\,\varphi)\right)\vec{\jmath} + \frac{d}{dt}(cos\,\theta)\;\vec{k}\right]$$

Also,

$$\vec{V} = \dot{r}\,\vec{U}_r + r \cdot \left[\left(\dot{\theta} \cdot \cos\theta \cdot cos\,\varphi + sin\,\theta \cdot (\dot{\varphi}(-\sin\varphi))\right)\vec{\imath} + \left(\dot{\theta} \cdot cos\,\theta \cdot sin\,\varphi + sin\,\theta \cdot (\dot{\varphi}\cos\varphi)\right)\vec{\jmath} + \dot{\theta}(-\,sin\,\theta)\,\vec{k}\right]$$

Simplifying this expression leads to:

$$\vec{V} = \dot{r}\,\vec{U}_r + r \cdot [(\dot{\theta} \cdot \cos\theta \cdot cos\,\varphi - \dot{\varphi} \cdot sin\,\theta \cdot \sin\varphi)\vec{\imath} + (\dot{\theta} \cdot cos\,\theta \cdot sin\,\varphi + \dot{\varphi} \cdot sin\,\theta \cdot cos\,\varphi)\,\vec{\jmath} - \dot{\theta}\,\sin\theta\;\vec{k}]$$

Rearranging the last we obtain:

$$\vec{V} = \dot{r}\,\vec{U_r} + r\cdot\dot{\varphi}\cdot \sin\theta \cdot \underbrace{\left(-\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath}\right)}_{=\overrightarrow{U_\varphi}}$$

$$+\, r\cdot\dot{\theta}\cdot \underbrace{\left(\cos\theta\cdot\cos\varphi\ \vec{\imath} + \cos\theta\cdot\sin\varphi\ \vec{\jmath}\ - \sin\theta\ \vec{k}\right)}_{=\overrightarrow{U_\theta}}$$

At the end:

$$\vec{V} = \dot{r}\,\vec{U_r} + r\cdot\dot{\theta}\cdot\overrightarrow{U_\theta} + r\cdot\dot{\varphi}\cdot\sin\theta\cdot\overrightarrow{U_\varphi}$$

Therefore, the components of the velocity vector ($\vec{V}$) are:

$$V_r = \dot{r}$$
$$V_\theta = r\cdot\dot{\theta}$$
$$V_\varphi = r\cdot\dot{\varphi}\cdot\sin\theta$$

These components represent the radial, polar, and azimuthal components of the velocity vector in spherical coordinates, respectively.

✓ **Acceleration vector ($\vec{a}$)**

The acceleration vector in spherical coordinates is expressed as follows:

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2}{dt^2}\left(r\overrightarrow{U_r}\right) = \left(\frac{d^2 r}{dt^2}\right)\vec{U_r} + r\cdot\frac{d^2\overrightarrow{U_r}}{dt^2}$$

Or recall the expression for the velocity vector and differentiate it with respect to time:

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\dot{r}\,\vec{U_r} + r\cdot\dot{\varphi}\cdot\sin\theta\cdot\overrightarrow{U_\varphi} + \ r\cdot\dot{\theta}\cdot\overrightarrow{U_\theta}\right)$$

Distributing the derivation gave the following expression:

$$\vec{a} = \frac{d}{dt}\left(\dot{r}\,\vec{U_r}\right) + \frac{d}{dt}\left(r\cdot\dot{\varphi}\cdot\sin\theta\cdot\overrightarrow{U_\varphi}\right) + \frac{d}{dt}\ \left(r\cdot\dot{\theta}\cdot\overrightarrow{U_\theta}\right)$$

Let's differentiate each term separately:

➢ **First term:**

$$\frac{d}{dt}\left(\dot{r}\,\vec{U_r}\right) = \frac{d}{dt}(\dot{r}\,)\cdot\vec{U_r} + \dot{r}\cdot\frac{d}{dt}\left(\vec{U_r}\right)$$

$$\frac{d}{dt}\left(\dot{r}\,\vec{U_r}\right) = \ddot{r}\cdot\vec{U_r} + \dot{r}\cdot\dot{\varphi}\cdot\sin\theta\,\overrightarrow{U_\varphi} + \dot{r}\cdot\dot{\theta}\,\overrightarrow{U_\theta}$$

- **Second term:**

$$\frac{d}{dt}\left(r\cdot\dot{\varphi}\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}\right)=\frac{d}{dt}(r)\cdot\dot{\varphi}\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}+r\cdot\frac{d}{dt}(\dot{\varphi})\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}$$

$$+r\cdot\dot{\varphi}\cdot\frac{d}{dt}(sin\,\theta)\cdot\overrightarrow{U_\varphi}+r\cdot\dot{\varphi}\cdot sin\,\theta\cdot\frac{d}{dt}\left(\overrightarrow{U_\varphi}\right)$$

Let's expand the last term:

$$\frac{d}{dt}\left(\overrightarrow{U_\varphi}\right)=\frac{d}{dt}\ (-\,sin\,\varphi\ \vec{\imath}+cos\,\varphi\ \vec{\jmath})=-\dot{\varphi}\cdot(\cos\varphi\ \vec{\imath}+\sin\varphi\ \vec{\jmath})$$

Now substitute this back into the expression of the second term:

$$\frac{d}{dt}\left(r\cdot\dot{\varphi}\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}\right)=\dot{r}\cdot\dot{\varphi}\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}+r\cdot\ddot{\varphi}\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}$$

$$+\,r\cdot\dot{\varphi}\cdot\left(\dot{\theta}\cdot\cos\theta\right)\cdot\overrightarrow{U_\varphi}+r\cdot\dot{\varphi}\cdot sin\,\theta\cdot\left(-\dot{\varphi}\cdot(\cos\varphi\,\vec{\imath}+\sin\varphi\,\vec{\jmath})\right)$$

Moreover:

$$\frac{d}{dt}\left(r\cdot\dot{\varphi}\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}\right)=\left(\dot{r}\cdot\dot{\varphi}\cdot sin\,\theta+r\cdot\ddot{\varphi}\cdot sin\,\theta+\ r\cdot\dot{\varphi}\cdot\dot{\theta}\cdot cos\,\theta\right)\overrightarrow{U_\varphi}$$

$$-\,r\cdot\dot{\varphi}^2\cdot sin\,\theta\cdot(\cos\varphi\,\vec{\imath}+\sin\varphi\,\vec{\jmath})$$

As we can see, cartesian unit vectors $\vec{\imath}$ and $\vec{\jmath}$ appear in the back expression. So, to express them in terms of spherical coordinates, we can use the inverse of the following orthogonal matrix:

$$\begin{pmatrix}\overrightarrow{U_r}\\ \overrightarrow{U_\theta}\\ \overrightarrow{U_\varphi}\end{pmatrix}=\begin{pmatrix}sin\,\theta\cdot cos\,\varphi & sin\,\theta\cdot sin\,\varphi & cos\,\theta\\ cos\,\varphi\cdot cos\,\theta & sin\,\varphi\cdot cos\,\theta & -\,sin\,\theta\\ -\sin\varphi & \cos\varphi & 0\end{pmatrix}\times\begin{pmatrix}\vec{\imath}\\ \vec{\jmath}\\ \vec{k}\end{pmatrix}$$

The inverse of this matrix is expressed as follows:

$$\begin{pmatrix}\vec{\imath}\\ \vec{\jmath}\\ \vec{k}\end{pmatrix}=\begin{pmatrix}sin\,\theta\cdot cos\,\varphi & cos\,\theta\cdot cos\,\varphi & -sin\,\varphi\\ sin\,\theta\cdot sin\,\varphi & cos\,\theta\cdot sin\,\varphi & cos\,\varphi\\ cos\,\theta & -\,sin\,\theta & 0\end{pmatrix}\times\begin{pmatrix}\overrightarrow{U_r}\\ \overrightarrow{U_\theta}\\ \overrightarrow{U_\varphi}\end{pmatrix}$$

As a consequence:

$$\vec{\imath}=sin\,\theta\cdot cos\,\varphi\ \overrightarrow{U_r}+cos\,\theta\cdot cos\,\varphi\ \overrightarrow{U_\theta}-sin\,\varphi\,\overrightarrow{U_\varphi}$$

$$\vec{\jmath}=sin\,\theta\cdot sin\,\varphi\ \overrightarrow{U_r}+cos\,\theta\cdot sin\,\varphi\ \overrightarrow{U_\theta}+\ cos\,\varphi\ \overrightarrow{U_\varphi}$$

$$\vec{k}=cos\,\theta\ \overrightarrow{U_r}-sin\,\theta\ \overrightarrow{U_\theta}$$

Now, let's replace $\vec{\imath}$ and $\vec{\jmath}$ in the expression second term

$$\frac{d}{dt}\left(r\cdot\dot{\varphi}\cdot sin\,\theta\cdot\overrightarrow{U_\varphi}\right)=-r\cdot\dot{\varphi}^2\cdot\sin^2\theta\ \overrightarrow{U_r}\ -\ r\cdot\dot{\varphi}^2\cdot sin\,\theta\,cos\,\theta\ \overrightarrow{U_\theta}$$
$$+\left(\dot{r}\cdot\dot{\varphi}\cdot sin\,\theta+r\cdot\ddot{\varphi}\cdot sin\,\theta+\ r\cdot\dot{\varphi}\cdot\dot{\theta}\cdot cos\,\theta\right)\overrightarrow{U_\varphi}$$

- **Third term:**

Starting from the expression:

$$\frac{d}{dt}\left(r\cdot\dot{\theta}\cdot\overrightarrow{U_\theta}\right)=\frac{d}{dt}\,(r)\cdot\dot{\theta}\cdot\overrightarrow{U_\theta}+r\cdot\frac{d}{dt}\,(\dot{\theta})\cdot\overrightarrow{U_\theta}+r\cdot\dot{\theta}\cdot\frac{d}{dt}\,(\overrightarrow{U_\theta})$$

This term can be expressed as follows:

$$\frac{d}{dt}\left(r\cdot\dot{\theta}\cdot\overrightarrow{U_\theta}\right)=\dot{r}\cdot\dot{\theta}\cdot\overrightarrow{U_\theta}+r\cdot\ddot{\theta}\cdot\overrightarrow{U_\theta}+r\cdot\dot{\theta}\cdot\frac{d}{dt}\,(\overrightarrow{U_\theta})$$

Let's develop the differentiation of the polar unit vector $(\overrightarrow{U_\theta})$ with respect to time:

$$\frac{d}{dt}\,(\overrightarrow{U_\theta})=\frac{d}{dt}\left(cos\,\varphi\cdot cos\,\theta\ \vec{\imath}+sin\,\varphi\cdot cos\,\theta\ \vec{\jmath}-sin\,\theta\ \vec{k}\right)$$

$$\frac{d}{dt}\,(\overrightarrow{U_\theta})=\frac{d}{dt}(cos\,\varphi\cdot cos\,\theta)\,\vec{\imath}+\frac{d}{dt}(\sin\varphi\cdot cos\,\theta)\,\vec{\jmath}-\frac{d}{dt}(sin\,\theta)\,\vec{k}$$

Now, let's distribute the differentiation:

$$\frac{d}{dt}\,(\overrightarrow{U_\theta})=\left[\frac{d}{dt}(cos\,\varphi)\cdot cos\,\theta+cos\,\varphi\cdot\frac{d}{dt}(\cos\theta)\right]\vec{\imath}$$
$$+\left[\frac{d}{dt}(\sin\varphi)\cdot cos\,\theta+sin\,\varphi\cdot\frac{d}{dt}(\cos\theta)\right]\vec{\jmath}-\frac{d}{dt}(sin\,\theta)\,\vec{k}$$

So, the complete expression for $\frac{d}{dt}\,(\overrightarrow{U_\theta})$ is:

$$\frac{d}{dt}\,(\overrightarrow{U_\theta})=\left[(-\dot{\varphi}\cdot\sin\varphi)\cdot cos\,\theta+cos\,\varphi\cdot\left(-\dot{\theta}\cdot\sin\theta\right)\right]\vec{\imath}$$
$$+\left[\dot{\varphi}\cdot\cos\varphi\cdot cos\,\theta+sin\,\varphi\cdot\left(-\dot{\theta}\cdot sin\,\theta\right)\right]\vec{\jmath}-\dot{\theta}\cdot\cos\theta\ \vec{k}$$

Now, let's simplify further:

$$\frac{d}{dt}\,(\overrightarrow{U_\theta})=\left[-\dot{\varphi}\cdot sin\,\varphi\cdot cos\,\theta-\dot{\theta}\cdot cos\,\varphi\cdot sin\,\theta\right]\vec{\imath}$$
$$+\left[\dot{\varphi}\cdot\cos\varphi\cdot cos\,\theta-\dot{\theta}\cdot sin\,\varphi\cdot sin\,\theta\right]\vec{\jmath}-\dot{\theta}\cdot\cos\theta\ \vec{k}$$

Rearranging the last expression leads to:

$$\frac{d}{dt}\left(\overrightarrow{U_\theta}\right) = \dot{\varphi} \cdot \cos\theta \cdot \underbrace{\left(-\sin\varphi\ \vec{\imath} + \cos\varphi\ \vec{\jmath}\right)}_{=\overrightarrow{U_\varphi}}$$

$$+\,\dot{\theta} \cdot \underbrace{\left(\cos\varphi \cdot \sin\theta\ \vec{\imath} + \sin\varphi \cdot \sin\theta\ \vec{\jmath} + \cos\theta\ \vec{k}\right)}_{=\overrightarrow{U_r}}$$

At the end the differentiation of the polar unit vector $(\overrightarrow{U_\theta})$ is articulated as follows

$$\frac{d}{dt}\left(\overrightarrow{U_\theta}\right) = \dot{\varphi} \cdot \cos\theta\, \overrightarrow{U_\varphi} + \dot{\theta} \cdot \overrightarrow{U_r}$$

Now let's replace this back in the expression of the third term:

$$\frac{d}{dt}\left(r \cdot \dot{\theta} \cdot \overrightarrow{U_\theta}\right) = \dot{r} \cdot \dot{\theta} \cdot \overrightarrow{U_\theta} + r \cdot \ddot{\theta} \cdot \overrightarrow{U_\theta} + r \cdot \dot{\theta} \cdot \dot{\varphi} \cdot \cos\theta\, \overrightarrow{U_\varphi} + r \cdot \dot{\theta}^2 \cdot \overrightarrow{U_r}$$

Now, let's simplify moreover:

$$\frac{d}{dt}\left(r \cdot \dot{\theta} \cdot \overrightarrow{U_\theta}\right) = r \cdot \dot{\theta}^2 \cdot \overrightarrow{U_r} + \left(\dot{r} \cdot \dot{\theta} + r \cdot \ddot{\theta}\right) \overrightarrow{U_\theta} + r \cdot \dot{\theta} \cdot \dot{\varphi} \cdot \cos\theta\, \overrightarrow{U_\varphi}$$

After being rearranged, the third term becomes:

$$\frac{d}{dt}\left(r \cdot \dot{\theta} \cdot \overrightarrow{U_\theta}\right) = -r \cdot \dot{\theta}^2\, \overrightarrow{U_r} + r \cdot \dot{\theta} \cdot \dot{\varphi} \cdot \cos\theta\, \overrightarrow{U_\varphi} + \left(\dot{r} \cdot \dot{\theta} + r \cdot \ddot{\theta}\right) \overrightarrow{U_\theta}$$

The addition of the three terms lead to the determination of the acceleration expression:

$$\vec{a} = \ddot{r} \cdot \overrightarrow{U_r} + \dot{r} \cdot \dot{\varphi} \cdot \sin\theta\, \overrightarrow{U_\varphi} + \dot{r} \cdot \dot{\theta}\, \overrightarrow{U_\theta} - r \cdot \dot{\varphi}^2 \cdot \sin^2\theta\, \overrightarrow{U_r} - r \cdot \dot{\varphi}^2 \cdot \sin\theta \cdot \cos\theta\, \overrightarrow{U_\theta}$$

$$+ \left(\dot{r} \cdot \dot{\varphi} \cdot \sin\theta + r \cdot \ddot{\varphi} \cdot \sin\theta + r \cdot \dot{\theta} \cdot \dot{\varphi} \cdot \cos\theta\right) \overrightarrow{U_\varphi} - r \cdot \dot{\theta}^2\, \overrightarrow{U_r}$$

$$+ r \cdot \dot{\theta} \cdot \dot{\varphi} \cdot \cos\theta\, \overrightarrow{U_\varphi} + \left(\dot{r} \cdot \dot{\theta} + r \cdot \ddot{\theta}\right) \overrightarrow{U_\theta}$$

The rearranging of this expression lead to the formula of the acceleration vector $(\vec{a})$ in spherical coordinates:

$$\vec{a} = \left(\ddot{r} - r \cdot \dot{\theta}^2 - r \cdot \dot{\varphi}^2 \cdot \sin^2\theta\right) \overrightarrow{U_r} + \left(r \cdot \ddot{\theta} + 2\,\dot{r} \cdot \dot{\theta} - r \cdot \dot{\varphi}^2 \cdot \sin\theta \cdot \cos\theta\right) \overrightarrow{U_\theta}$$

$$+ \left(r \cdot \ddot{\varphi} \cdot \sin\theta + 2\,\dot{r} \cdot \dot{\varphi} \cdot \sin\theta + 2\,r \cdot \dot{\theta} \cdot \dot{\varphi} \cdot \cos\theta\right) \overrightarrow{U_\varphi}$$

Therefore, the components of the acceleration vector $(\vec{a})$ are:

$$a_r = \ddot{r} - r \cdot \dot{\theta}^2 - r \cdot \dot{\varphi}^2 \cdot \sin^2\theta$$

$$a_\theta = r \cdot \ddot{\theta} + 2\,\dot{r} \cdot \dot{\theta} - r \cdot \dot{\varphi}^2 \cdot \sin\theta \cdot \cos\theta$$

$$a_\varphi = r \cdot \ddot{\varphi} \cdot \sin\theta + 2\,\dot{r} \cdot \dot{\varphi} \cdot \sin\theta + 2\,r \cdot \dot{\theta} \cdot \dot{\varphi} \cdot \cos\theta$$